9/10 Recitation

1. If the following statement is true, prove it; if it is false, provide a counterexample. If A and B are invertible $n \times n$ matrices, then so is $A + B$.

   False. Consider I_n and $-I_n$; both are invertible since they have n pivots. However, $I + (-I)$ is the zero matrix, which is not invertible since it has no pivots.

2. Use the Gauss-Jordan algorithm to find the inverse of $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap R1 and R2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Replace R2 by R2}-2\text{R1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Swap R2 and R3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & -2 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\text{Replace R3 with R3+3R1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & -2 & 3 \end{array}\right]$$

$$\xrightarrow{\text{Multiply R3 by } \frac{1}{4}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$$\xrightarrow[\text{Replace R2 by R2-2R3}]{\text{Replace R1 by R1-R3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$$\xrightarrow{\text{Replace R1 by R1-2R2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

Hence the inverse is $\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$

3. Find a 2×2 matrix that its its own inverse (other than the identity).

   The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is its own inverse since $AA = I$.

4. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations given by

$$S\Big(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\Big) = \begin{bmatrix} x+y \\ y+z \\ z \end{bmatrix} \qquad \text{and} \qquad T\Big(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\Big) = \begin{bmatrix} x-y+z \\ y-z \\ z \end{bmatrix}.$$

   (a) Show that S and T are inverses by showing $(S \circ T)(\mathbf{v}) = \mathbf{v}$ and $(T \circ S)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

   Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then

$$S(T(\mathbf{v})) = S\Big(\begin{bmatrix} v_1 - v_2 + v_3 \\ v_2 - v_3 \\ v_3 \end{bmatrix}\Big) = \begin{bmatrix} (v_1 - v_2 + v_3) + (v_2 - v_3) \\ (v_2 - v_3) + v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{v}$$

$$T(S(\mathbf{v})) = T\Big(\begin{bmatrix} v_1 + v_2 \\ v_2 + v_3 \\ v_3 \end{bmatrix}\Big) = \begin{bmatrix} (v_1 + v_2) - (v_2 + v_3) + v_3 \\ (v_2 + v_3) - v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{v}$$

   (b) Let A be the matrix associated to S and let B be the matrix associated to T. Write down A and B and verify that A and B are inverses of each other.

   $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. A quick check by hand or using Julia shows that $AB = BA = I$. Hence A and B are inverses.

5. Recall that $(AB)^T = B^T A^T$.

   (a) Verify this property is true for the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$.

   $(AB)^T = \begin{bmatrix} 1 & 4 \\ 1 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}$ and $B^T A^T = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}$.

   (b) Let A be an $m \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$. Show that $(A\mathbf{v})^T = \mathbf{v}^T A^T$.

Denote the $(i,j)-$entry of A by a_{ij} and let $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Then

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

Hence the i-th entry of $(A\mathbf{v})^T$, which is a row vector, is $a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$. On the other hand,

$$\mathbf{v}^T A^T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} A^T = \begin{bmatrix} v_1a_{11} + v_2a_{12} + \cdots + v_na_{1n} & \cdots & v_1a_{m1} + v_2a_{m2} + \cdots + v_na_{mn} \end{bmatrix}$$

which has i-th entry $a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$. Since $(A\mathbf{v})^T$ and $\mathbf{v}^T A^T$ have the same exact entries, we have that $(A\mathbf{v})^T = \mathbf{v}^T A^T$.

(c) Now let's prove the general property. Fill in the blanks.

Let $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$. Then

$$(AB)^T = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_n \end{bmatrix}^T = \begin{bmatrix} (A\mathbf{b}_1)^T \\ \vdots \\ (A\mathbf{b}_n)^T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^T A^T \\ \vdots \\ \mathbf{b}_n^T A^T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix} A^T = B^T A^T$$

6. Prove that $(A^T)^{-1} = (A^{-1})^T$.

Since $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ and $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, we have that $(A^{-1})^T$ is the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.